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# Tensor-like technique and quantum algebra $su_q(3)$

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**Abstract.** A new method is presented for constructing the irreducible representation and for calculating the Clebsch–Gordan Coefficient of the quantum algebra $su_q(3)$. The method is different from others and does not depend on any concrete realization of $su_q(3)$. This means that the result holds in general. A set of recurrence formula has been obtained for calculating the Clebsch–Gordan coefficient. In fact the recurrence formula is aimed at the so-called scalar factor of $su_q(3)$, and means that the Racah factorization lemma also holds for the quantum algebra $su_q(3)$.

## 1. Introduction

In a previous paper [1], hereafter referred to as part I, we developed a new technique to construct the irreducible representations (irreps) of the quantum algebra $su_q(3)$ explicitly. The method is based on the fact that the generators of $su_q(3)$ are written as $J_0, J_\pm \in su_q(2)$ and $T_{\pm 1/2}$, $V_{\pm 1/2}$. The latter, satisfying Serre and Serre-like relations, are considered as $\frac{1}{2}$ rank tensor-like operators of $su_q(2)$, as is done in the classical Lie algebra [2]. Thus their matrix elements can be easily derived in Elliott-like basis vectors, and a set of recurrence formula for the reduced matrix elements has been obtained.

However, in [1] we make use of a boson realization of $su_q(3)$ algebra, concerned to deduce certain algebraic relations which are subsequently exploited to obtain the key results. Here, we first prove that the main results in [1] do not depend on any concrete realizations, so that the results hold in general.

Second, we also derive the Clebsch–Gordan coefficient (CGC) of $su_q(3)$ algebra with the help of the above technique. Ma [3] has also considered the same problem, but only gave a few of numerical tables. We will derive a recurrence relation. From this relation we calculate all of the CGC of $su_q(3)$ including Ma's. In fact, our formula is aimed at the so-called scalar factor (SF) which is invariant under algebra $su_q(2)$. This means that the CGC of $su_q(3)$ can be factorized by the generalized Racah's factorization lemma just as for the classical Lie algebra su(3). Recently we have noticed similar considerations in the literature [4, 5].

This paper is organized as follows. In section 2, we rewrite the quantum algebra $su_q(3)$. Differing from [1] we shall not use the boson realization $su_q(3)$ algebra for obtaining all of the key results here. We have revised some errors which appeared in [1], thus the necessity for rewriting the $su_q(3)$ algebra in this section. In section 3, we calculate the CGC of $su_q(3)$. To do this we have to modify the action on the tensor product space and extend Racah's factorization lemma for the usual Lie algebra. From these results we give a recurrence formula for SF. Finally we give the main conclusions.

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## 2. The quantum algebra $su_q(3)$

The quantum algebra $su_q(3)$ is generated by $h_i$, $e_{\pm i}$ ($h_i^+ = h_i$, $e_i^+ = e_{-i}$, $i = 1, 2$) and obeys the relations [6]

$$[h_i, e_{\pm j}] = \pm a_{ij} e_{\pm j} \qquad i, j = 1, 2 \tag{1a}$$

$$[e_i, e_{-j}] = \delta_{ij}[h_j] \qquad i, j = 1, 2 \tag{1b}$$

and Serre's relations

$$e_{\pm 1}^2 e_{\pm 2} + e_{\pm 2} e_{\pm 1}^2 = [2] e_{\pm 1} e_{\pm 2} e_{\pm 1} \tag{1c}$$

where

$$(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

$[x] = (q^x - q^{-x})/(q - q^{-1})$ and $x$ is a number or an operator.

In this algebra, two additional operators are defined usually

$$e_3 = (\mathrm{ad} e_1)_q e_2 = e_1 e_2 - q e_2 e_1 \tag{2a}$$

$$e_{-3} = (\mathrm{ad} e_{-2})_q e_{-1} = e_{-2} e_{-1} - q^{-1} e_{-1} e_{-2}. \tag{2b}$$

Notice that

$$e_3^+ = (e_{-3})_{\bar{q}^{-1}}. \tag{3}$$

Here and below, the symbol $(x)_{\bar{q}^{-1}}$ means $q \to \bar{q}^{-1}$ in the operator $x$, where $\bar{q}$ is a complex conjugate of $q$. Making use of (1), we obtain

$$[h_i, e_{\pm 3}] = \pm e_{\pm 3} \qquad i = 1, 2 \tag{4a}$$

$$[e_3, e_{-3}] = [h_1 + h_2] \tag{4b}$$

and

$$q^{-1} e_{\mp 1}^2 e_{\pm 3} + q e_{\pm 3} e_{\mp}^2 = [2] e_{\mp 1} e_{\pm 3} e_{\mp 1}. \tag{5}$$

The relations can be called Serre-like relation as in [1].

We now redefine the generators

$$J_0 = h_1/2 \qquad J_\pm = e_{\pm 1} \tag{6a}$$

$$Q = -(h_1 + 2h_2) \tag{6b}$$

and

$$T_{1/2} = -e_{-2} \qquad V_{-1/2} = e_2 \tag{6c}$$

and introduce two auxiliary operators

$$T_{-1/2} = q^{-h_2} e_{-3} \qquad V_{1/2} = q^{h_2 - 1} e_3. \tag{7}$$

From (6c) and (7) we have

$$V_{-1/2}^+ = -T_{1/2} \qquad V_{1/2}^+ = (T_{-1/2})_{\bar{q}^{-1}}. \tag{8}$$

From (1)–(8), obviously the following relations are satisfied

$$[Q, J_0] = [Q, J_\pm] = 0 \tag{9a}$$

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0] \tag{9b}$$

$$[J_0, T_s]=sT_s \qquad [J_0, V_s]=sV_s \qquad s=\pm 1/2 \tag{9c}$$

$$[Q, T_s]=3T_s \qquad [Q, V_s]=-3V_s \qquad s=\pm 1/2 \tag{9d}$$

and

$$J_\pm^2 T_{\mp 1/2}+T_{\mp 1/2}J_\pm^2=[2]J_\pm T_{\mp 1/2}J_\pm \tag{10a}$$

$$J_\pm^2 V_{\mp 1/2}+V_{\mp 1/2}J_\pm^2=[2]J_\pm V_{\mp 1/2}J_\pm . \tag{10b}$$

When $q$ is not a root of unity, the finite-dimensional representations of $\mathrm{su}_q(3)$ are given by the integer $\lambda$ and $\mu$ as classical Lie algebra [3]. Let $|(\lambda\mu)\varepsilon jm\rangle$ be orthonormal bases in representation space $L$, which are Elliott-like and defined by

$$\langle(\lambda\mu)\varepsilon' j' m'|Q|(\lambda\mu)\varepsilon jm\rangle=\varepsilon\delta_{\varepsilon'\varepsilon}\delta_{j'j}\delta_{m'm} \tag{11a}$$

$$\langle(\lambda\mu)\varepsilon' j' m'|J_0|(\lambda\mu)\varepsilon jm\rangle=m\delta_{\varepsilon'\varepsilon}\delta_{j'j}\delta_{m'm} \tag{11b}$$

$$\langle(\lambda\mu)\varepsilon' j' m'|J_\pm|(\lambda\mu)\varepsilon jm\rangle=([j\mp m][j\pm m+1])^{1/2}\delta_{\varepsilon'\varepsilon}\delta_{j'j}\delta_{m'm\pm 1} . \tag{11c}$$

From (5) we obtain an equation as (12) in [1]. Solving the equation we have

$$\langle\varepsilon+3, j'm'|T_s|\varepsilon jm\rangle=\frac{\langle\varepsilon+3j'\|T\|\varepsilon j\rangle}{\sqrt{[2j'+1]}}\, q^{A/2}C_q(jm\tfrac{1}{2}s|j'm') \tag{12a}$$

where

$$A=(-1)^{s+j-j'}(j+1/2)-m' \qquad j'=j\pm\tfrac{1}{2} \qquad s=\pm\tfrac{1}{2}. \tag{12b}$$

For simplicity we omit the quantum number $(\lambda\mu)$ from now on. $C_q(jm\frac{1}{2}s|j'm')$ is a CGC of $\mathrm{su}_q(2)$ [7–9]. The matrix element of $V_s$ is similar. Equation (12) is considered as a $q$-Wigner–Eckart theorem. Thus the operators $T_s$ and $V_s$ can be considered as $\frac{1}{2}$ rank tensor-like operators of $\mathrm{su}_q(2)$.

With the help of (8) and the following relations

$$J_+^2(V_{1/2})^+ + (V_{1/2})^+ J_+^2=[2]J_+(V_{1/2})^+ J_+ \tag{13a}$$

$$J_+^2(T_{-1/2})_{\bar{q}^{-1}}+(T_{-1/2})_{\bar{q}^{-1}}J_+^2=[2]J_+(T_{-1/2})_{\bar{q}^{-1}}J_+ \tag{13b}$$

we can derive

$$\begin{aligned}\langle\varepsilon j\|V\|\varepsilon' j'\rangle^* &= (-1)^{1/2+j'-j}\langle\varepsilon' j'\|(T)_{\bar{q}^{-1}}\|\varepsilon j\rangle\\ &=(-1)^{1/2+j'-j}\langle\varepsilon' j'\|T\|\varepsilon j\rangle .\end{aligned} \tag{14}$$

Finally, as in [1], using (14) and

$$[T_{1/2}, V_{-1/2}]=-[Q/2+J_0] \tag{15a}$$

$$[T_{-1/2}, V_{1/2}]=[Q/2-J_0] \tag{15b}$$

we have

$$\begin{aligned}&|\langle\varepsilon_0-3(a+b), j_0+(a-b)/2\|T\|\varepsilon_0-3(a+b+1), j_0+(a-b+1)/2\rangle|^2\\ &\qquad=[1+a][2j_0+2+a][\varepsilon_0/2-j_0-a]\end{aligned} \tag{16a}$$

$$\begin{aligned}&|\langle\varepsilon_0-3(a+b), j_0+(a-b)/2\|T\|\varepsilon_0-3(a+b+1), j_0+(a-b-1)\rangle|^2\\ &\qquad=[1+b][2j_0-b][\varepsilon_0/2+j_0+1-b]\end{aligned} \tag{16b}$$

$$a=0, 1, 2, \ldots , \qquad b=0, 1, 2, \ldots . \tag{17}$$

In (6), $\varepsilon_0 = 2\lambda + \mu, j_0 = \mu/2$. $\varepsilon_0$ is a maximum value of $\varepsilon$, which is determined by $T_s|(\lambda\mu)\varepsilon_0 j_0 m\rangle = 0$. And $j_0$ is the only $j$ value corresponding to $\varepsilon_0$. In (17), $a$ and $b$ are related to $n$ and $i$ of (20) in [1] by $n = a + b$, $i = b$.

We can choose the phase factors from among the basis vectors in order that the $\langle \varepsilon' j' \| V \| \varepsilon j \rangle$ are real and positive, so the $\langle \varepsilon' j' \| T \| \varepsilon j \rangle$ can be fixed by (14). That is to say, the irreps can be completely determined by the present technique.

For instance:

(1) $(\lambda\mu) = (20)$ $\varepsilon_0 = 4$ $j_0 = 0$

**Table 1a.** The values of $\langle \varepsilon' j' \| V \| \varepsilon j \rangle$.

| $\varepsilon'$ | $j'$ | $\varepsilon$ | $j$ | $\langle \varepsilon' j' \| V \| \varepsilon j \rangle$ |
|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | 4 | 0 | [2] |
| −2 | 1 | 1 | $\frac{1}{2}$ | [2][3] |

**Table 1b.** The non-vanishing values of $Z = \langle \varepsilon' j' m' | V_{-1/2} | \varepsilon j m \rangle$.

| $\varepsilon'$ | $j'$ | $m'$ | $\varepsilon$ | $j$ | $m$ | $Z$ |
|---|---|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 4 | 0 | 0 | [2] |
| −2 | 1 | −1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | [2] |
| −2 | 1 | 0 | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 1 |

(2) $(\lambda\mu) = (21)$ $\varepsilon_0 = 5$ $j_0 = \frac{1}{2}$

**Table 2a.** The values of $\langle \varepsilon' j' \| V \| \varepsilon j \rangle$.

| $\varepsilon'$ | $j'$ | $\varepsilon$ | $j$ | $\langle \varepsilon' j' \| V \| \varepsilon j \rangle$ |
|---|---|---|---|---|
| 2 | 1 | 5 | $\frac{1}{2}$ | [2][3] |
| 2 | 0 | 5 | $\frac{1}{2}$ | [4] |
| −1 | $\frac{3}{2}$ | 2 | 1 | [2][4] |
| −1 | $\frac{1}{2}$ | 2 | 1 | [4] |
| −1 | $\frac{1}{2}$ | 2 | 0 | [2][3] |
| −4 | 1 | −1 | $\frac{3}{2}$ | [4] |
| −4 | 1 | −1 | $\frac{1}{2}$ | [2][4] |

**Table 2b.** The non-vanishing values of $Z = \langle \varepsilon' j' m' | V_{-1/2} | \varepsilon j m \rangle$.

| $\varepsilon'$ | $j'$ | $m'$ | $\varepsilon$ | $j$ | $m$ | $Z$ |
|---|---|---|---|---|---|---|
| 2 | −1 | −1 | −1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | [2] |
| | | 0 | | | $\frac{1}{2}$ | 1 |
| | 0 | 0 | | | $\frac{1}{2}$ | [4]/[2] |
| −1 | $\frac{3}{2}$ | $-\frac{3}{2}$ | 2 | 1 | −1 | [2] |
| | | $-\frac{1}{2}$ | | | 0 | [2]/[3] |
| | | $\frac{1}{2}$ | | | 1 | [2]/[3] |
| | $\frac{1}{2}$ | $\frac{1}{2}$ | | | 1 | [4]/[3] |
| | | $-\frac{1}{2}$ | | | 0 | [4]/[2][3] |
| | | $-\frac{1}{2}$ | | 0 | 0 | [3] |
| −4 | 1 | 1 | −1 | $\frac{3}{2}$ | $-\frac{1}{2}$ | 1 |
| | | 0 | | | $\frac{1}{2}$ | [2]/[3] |
| | | −1 | | | $-\frac{1}{2}$ | 1/[3] |
| | | −1 | | $\frac{1}{2}$ | $-\frac{1}{2}$ | [2][4]/[3] |
| | | 0 | | | $\frac{1}{2}$ | [4]/[3] |

These results are the same as those given by Ma [3], perhaps indicating that our choice for $T_{-1/2}$ and $V_{1/2}$ is suitable.

With the help of the highest weight vector $|(\lambda\mu)\varepsilon_0 j_0 m\rangle$ all of the basis vectors $|(\lambda\mu)\varepsilon jm\rangle$ can be obtained since

$$|(\lambda\mu)\varepsilon-3j'm'\rangle=(-1)^{1/2+j-j'}N(\varepsilon j'j)\sum_{ms}C_q(jm\tfrac{1}{2}s|j'm')V_s|(\lambda\mu)\varepsilon jm\rangle \tag{18a}$$

where $N(\varepsilon j'j)$ is a normalized constant [1]

$$\{N(\varepsilon j'j)\}^{-1}=\langle\varepsilon j\|T\|\varepsilon-3j'\rangle/\sqrt{[2j'+1]}\sum_{ms}\{C_q(jm\tfrac{1}{2}s|j'm')\}^2 q^{A/2}. \tag{18b}$$

## 3. The Clebsch–Gordan coefficient of $su_q(3)$

The Clebsch–Gordan coefficients appear in the tensor product space $L\otimes L$. As in the classical case, it is now necessary to define the action of the generators on the space $L\otimes L$. Here we try the usual definition for the operators $H=Q, J_0$ or their linear combinations

$$H(f\otimes g)=Hf\otimes g+f\otimes Hg \qquad f\otimes g\in L\otimes L$$

We write, when $H$ act on $L\otimes L$

$$\Delta(H)=H\otimes 1+1\otimes H. \tag{19}$$

The map $\Delta$ is called the coproduct and defined in the Hopf algebra [4]. From (19) it follows that

$$\Delta(q^{aH})=q^{aH}\otimes q^{aH}. \tag{20}$$

$\Delta(T_{1/2})$ and $\Delta(V_{-1/2})$ should be defined in such a way that they are a homomorphism of the algebra $su_q(3)$ into $su_q(3)\otimes su_q(3)$. In particular, we require that

$$[\Delta(T_{1/2}),\Delta(V_{-1/2})] =(q^{-(Q/2+J_0)}\otimes q^{-(Q/2+J_0)}-q^{(Q/2+J_0)}\otimes q^{(Q/2+J_0)})/(q-q^{-1}). \tag{21}$$

We find that a definition similar to (19) is not compatible with (21), and instead, we have to define

$$\Delta(T_{1/2})=T_{1/2}\otimes q^{1/2(Q/2+J_0)}+q^{-1/2(Q/2+J_0)}\otimes T_{1/2} \tag{22a}$$

$$\Delta(V_{-1/2})=V_{-1/2}\otimes q^{1/2(Q/2+J_0)}+q^{-1/2(Q/2+J_0)}\otimes V_{-1/2}. \tag{22b}$$

Note that the coproducts of the auxiliary operators $T_{-1/2}$ and $V_{1/2}$ are very complex, but they will not appear in our discussion below; accordingly we will not write them here. This is another superior point of our method.

Now let the orthonormal basis $|\alpha(\lambda\mu)\varepsilon jm\rangle$ of $L\otimes L$ be

$$|\alpha(\lambda\mu)\varepsilon jm\rangle=\sum_{\varepsilon_a j_a \varepsilon_b j_b}\begin{pmatrix}(\lambda_a\mu_a & (\lambda_b\mu_b) \\ \varepsilon_a j_a & \varepsilon_b j_b\end{pmatrix}\left\|\begin{matrix}(\lambda\mu)\alpha \\ \varepsilon j\end{matrix}\right)|\varepsilon_a j_a,\varepsilon_b j_b;\varepsilon jm\rangle \tag{23}$$

where $\alpha$ labels multiplicity of the reduction $(\lambda_a\mu_a)\otimes(\lambda_b\mu_b)\to(\lambda\mu)$.

$$\begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) & \| & (\lambda\mu)\alpha \\ \varepsilon_a j_a & \varepsilon_b j_b & \| & \varepsilon j \end{pmatrix}$$

is called the scalar factor (SF) of $su_q(3)$, which is invariant for the algebra $su_q(2)$, and $|\varepsilon_a j_a, \varepsilon_b j_b; \varepsilon jm\rangle$ is defined as

$$|\varepsilon_a j_a, \varepsilon_b j_b; \varepsilon jm\rangle = \sum_{m_a m_b} C_q(j_a m_a j_b m_b | jm) |(\lambda_a\mu_a)\varepsilon_a j_a m_a\rangle |(\lambda_b\mu_b)\varepsilon_b j_b m_b\rangle. \tag{24}$$

Hence

$$|\alpha(\lambda\mu)\varepsilon jm\rangle = \sum_{\substack{\varepsilon_a j_a m_a \\ \varepsilon_b j_b m_b}} \begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) & | & (\lambda\mu)\alpha \\ \varepsilon_a j_a m_a & \varepsilon_b j_b m_b & | & \varepsilon jm \end{pmatrix} |(\lambda_a\mu_a)\varepsilon_a j_a m_a\rangle |(\lambda_b\mu_b)\varepsilon_b j_b m_b\rangle. \tag{25}$$

That is to say that the CGC

$$\begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) & | & (\lambda\mu)\alpha \\ \varepsilon_a j_a m_a & \varepsilon_b j_b m_b & | & \varepsilon jm \end{pmatrix}$$

of $su_q(3)$ can be written as

$$\begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) & | & (\lambda\mu)\alpha \\ \varepsilon_a j_a m_a & \varepsilon_b j_b m_b & | & \varepsilon jm \end{pmatrix} = \begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) & \| & (\lambda\mu)\alpha \\ \varepsilon_a j_a & \varepsilon_b j_b & \| & \varepsilon j \end{pmatrix} C_q(j_a m_a j_b m_b | jm). \tag{26}$$

Equation (26) is considered as a generalized Racah factorization lemma.

In the following, we will give formulae for calculating SF values. Because

$$T_s |(\lambda\mu)\varepsilon_0 j_0 m\rangle = 0 \qquad s = \pm\tfrac{1}{2} \tag{27}$$

we can obtain

$$\sum_{\varepsilon_a j_a \varepsilon_b j_b} \langle \varepsilon'_a j'_a, \varepsilon'_b j'_b; \varepsilon' j' \| T \| \varepsilon_a j_a, \varepsilon_b j_b; \varepsilon_0 j_0\rangle \begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) & \| & (\lambda\mu)\alpha \\ \varepsilon_a j_a & \varepsilon_b j_b & \| & \varepsilon_0 j_0 \end{pmatrix} = 0. \tag{28}$$

Generally, if $\alpha \neq 1$, then there are several solutions to (28). Using (22) and the symmetrical properties of the CGC of $su_q(2)$, we obtain

$$\begin{aligned} &\langle \varepsilon'_a j'_a, \varepsilon'_b j'_b; \varepsilon' j' \| T \| \varepsilon_a j_a, \varepsilon_b j_b; \varepsilon j\rangle \\ &\quad = (-1)^{1/2+j'_a+j_b+j} \sqrt{[2j+1][2j'+1]} \begin{Bmatrix} \frac{1}{2} & j_a & j'_a \\ j_b & j' & j \end{Bmatrix}_q \\ &\qquad \times \langle \varepsilon_a + 3j'_a \| T^{(a)} \| \varepsilon_a j_a\rangle q^{B_1/2} \\ &\qquad + (-1)^{1/2+j'_b+j_a+j} \sqrt{[2j+1][2j'+1]} \begin{Bmatrix} \frac{1}{2} & j_b & j'_b \\ j_a & j' & j \end{Bmatrix}_q \\ &\qquad \times \langle \varepsilon_b + 3j'_b \| T^{(b)} \| \varepsilon_b j_b\rangle q^{B_2/2} \end{aligned} \tag{29}$$

where

$$B_1 = \varepsilon_b/2 - (-1)^{1/2+j_a-j'_a}(j_a+\tfrac{1}{2}) + (-1)^{1/2+j-j'}(j+\tfrac{1}{2})$$

$$B = -\varepsilon_a/2 + (-1)^{1/2+j_b-j'_b}(j_b+\tfrac{1}{2}) - (-1)^{1/2+j-j'}(j+\tfrac{1}{2}).$$

Here

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}_q$$

is a quantum Racah coefficient of $su_q(2)$ [7, 10]. And $\langle \varepsilon' j' \| T^{(a) \text{ or } (b)} \| \varepsilon j \rangle$ can be calculated using (16).

Using (18) and (23) we can finally obtain

$$\begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) \\ \varepsilon'_a j'_a & \varepsilon'_b j'_b \end{pmatrix}\left\| \begin{matrix} (\lambda\mu)\alpha \\ \varepsilon - 3j' \end{matrix} \right)$$

$$= \{\langle \varepsilon j \| T \| \varepsilon - 3j' \rangle\}^{-1}$$

$$= \sum_{\varepsilon_a j_a \varepsilon_b j_b} \begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) \\ \varepsilon_a j_a & \varepsilon_b j_b \end{pmatrix}\left\| \begin{matrix} (\lambda\mu)\alpha \\ \varepsilon j \end{matrix} \right) \langle \varepsilon_a j_a,\ \varepsilon_b j_b;\ \varepsilon j \| T \| \varepsilon'_a j'_a,\ \varepsilon'_b j'_b;\ \varepsilon' j' \rangle. \quad (30)$$

From (30) and the known

$$\begin{pmatrix} (\lambda_a\mu_a) & (\lambda_b\mu_b) \\ \varepsilon_a j_a & \varepsilon_b j_b \end{pmatrix}\left\| \begin{matrix} (\lambda\mu) \\ \varepsilon_0 j_0 \end{matrix} \right)$$

we can determine all of the SF and thus the CGC of $su_q(3)$. We have also calculated some numerical values including Ma's [3]. Our results are in agreement with [3]. Now we are planning to translate the method into a computer program.

In conclusion, the irreducible representations and CGC of quantum algebra $su_q(3)$ have been obtained by use of (18), which is a fundamental and key formula. We have also shown that Racah's factorization lemma can be extended to the case of quantum algebra $su_q(3)$, and that the SF can be calculated by our technique. Of course the present procedure is not suitable for the case $q$ being a root of unity. This case is still an open problem.

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